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A GENERAL EXTENSION OF FEUERBACH'S THEOREM

BY

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1. The interesting theorem in the geometry of the triangle named after Feuerbach* viz., that the nine points circle of a triangle touches its in- and ex-circles, has been shown to be a particular case of various other results. But many of the plane extensions of Feuerbach's result could be derived as particular cases of a more general theorem which, besides, has other important applications to the theory of the Orthopolar Circles of the triangle. The Theorem in question is:

THEOREM I:—*If Γ be a range† of conics and Γ' any other range formed by conics confocal with those of Γ, the mutual orthoptic circles of corresponding confocals in Γ and Γ' belong to a Coaxal system.*

2. It is well known that the totality of conics belonging to the net determined by three linearly independent conics can be put into (1-1) correspondence with the points of a projective plane π. Taking for the three linearly independent conics, the absolute conic Ω (consisting of the pair of circular points of the plane) and any two conics Σ_1, Σ_2 of the range Γ, it is evident that in this representation, the conic Ω will be represented by a point Ω and the conics of the range Γ by the points of the line, say Γ, joining the representative points Σ_1, Σ_2 in π. Now, a confocal system of conics is a range in which the absolute conic Ω is a member. Since the line joining any point Σ' in π to Ω will meet Γ in a point Σ, it follows that the points of the plane π correspond to the totality of conics confocal with those of Γ. Thus, any other line Γ' in π will represent a range of conics confocal with the conics of the range Γ. Further, a pair of corresponding confocals Σ in Γ and Σ' in Γ' will be represented in the π plane by points Σ, Σ' collinear with Ω and the point of intersection, say σ, of the lines Γ and Γ' will represent the self corresponding member σ in the two ranges.

* *Vide* Casey *Analytical Geometry*. pp. 125–6.

† A range of conics is a system touching four lines.

Let $\sum''$ be the harmonic conjugate of Ω w. r. t. the corresponding points $\sum$, $\sum'$. It is obvious that the locus of $\sum''$ is a line Γ'' through σ. Thus, the conics $\sum''$, which are the harmonic conjugates of the absolute Ω w. r. t. a pair of associated confocals like $\sum$, $\sum'$ belong to a range Γ'' to which also belongs the self corresponding member σ.

Now, the F-conic of two conics α, β passes through the points of intersection of the pairs of conics like $\alpha + \lambda\beta = 0$ and $\alpha - \lambda\beta = 0$, in the range determined by them. Thus the F-conic of Ω and $\sum''$ in the confocal system $(\sum, \sum')$ must pass through the common points of $\sum$ and $\sum'$. In other words, the circle passing through the intersections of the confocals $\sum$ and $\sum'$, which is their mutual orthoptic circle is the director-circle of $\sum''$. But, since the conics $\sum''$ belong to a range, their director circles are coaxal. It follows, therefore, that the mutual orthoptic circles of pairs of corresponding confocals belong to a coaxal system.

3. We next proceed to derive the well-known extensions of Feuerbach's theorem as special cases of Theorem I.

Let ABC be a triangle and Ω the absolute conic consisting of the pair of circular points. Now, the conics in the plane having ABC for a self polar triangle and out-polar to Ω belong to a pencil δ and are the rectangular hyperbolas through the in- and ex-centres of the triangle. Hence, any conic $\sum$ inscribed in the triangle is in-polar to all the conics of the pencil δ; so also is any conic $\sum'$ confocal with $\sum$.

If now, $\sum'$ is a point pair (P, P′) P and P′ are a pair of foci of $\sum$ and are conjugate points w. r. t. the conics of the pencil δ. In particular, P, P′ are conjugate w. r. t. the three line pairs of δ viz., the pairs of bisectors of the angles A, B, C. Thus, AP, AP′; BP, BP′; CP, CP′ are equally inclined to the bisectors of the angles A, B, C. Therefore P, P′ are isogonal conjugates w. r. t. the triangle. We have thus :

LEMMA I :—*A pair of isogonally conjugate points w. r. t. a triangle are conjugate w. r. t. the pencil of rectangular hyperbolas through the in- and ex-centres of the triangle.*

Let $\sum$ be any conic inscribed in the triangle and let its director circle cut the circumcircle of the triangle in X and X′. Since ABC is one triangle inscribed in the circumcircle and circumscribed to $\sum$, there will be an infinity of such triangles. Thus, if XY, XZ are

tangents to Σ through X meeting the circumcircle again in Y, Z then YZ touches Σ. Further, since X is a point on the director circle of Σ, XY and XZ are at right angles; therefore YZ passes through the circumcentre of ABC. Hence, corresponding to each point X or X′ we have a tangent to Σ through O. If, however, Σ passes through O, the two tangents to Σ through O coincide; Consequently the points X and X′ will also coincide so that the director circle of Σ touches the circumcircle of the triangle.

Now, the polar circle of the triangle is out-polar to any conic inscribed in the triangle and, therefore, cuts orthogonally its director circle. Hence the director circle of Σ is inverted into itself w. r. t. the polar circle. But the inverse of the circumcircle w. r. t. the polar circle is the nine points circle. Hence, the director circle of Σ touches also the nine points circle we have thus:

LEMMA II:—*If a conic inscribed in a triangle passes through the circumcentre, its director circle touches the nine points circle of the triangle.*

Consider a range Γ of conics inscribed in a triangle and let Γ' be any other range formed by conics confocal with those of Γ. Now, since the conics of the pencil δ are out-polar to those of the range Γ and to the absolute Ω, they are also out-polar to the conics of the range Γ'. In particular, the singular members of Γ' viz., the pairs of opposite vertices, say, X, X′; Y, Y′; Z, Z′ of the quadrilateral formed by their common tangents will be inpolar to the pencil δ. Thus X, X′; Y, Y′; Z, Z′ are by Lemma I, isogonal conjugates w. r. t. the triangle and are also the foci of the corresponding confocals in Γ. If we call a quadrilateral whose three pairs of opposite vertices are isogonal conjugates w. r. t. a triangle, an *isogonal quadrilateral* of the triangle, we infer that:

THEOREM II:—*With every range Γ' of conics inscribed in an isogonal quadrilateral of a triangle, it is possible to associate a unique range Γ of conics inscribed in the triangle such that to every conic Σ in Γ there corresponds a confocal Σ' in Γ'.*

Now, the mutual orthoptic circle of (X, X′) and the corresponding inscribed confocal, is the pedal circle of X, X′ w. r. t. the triangle. Hence we have by Theorem I and lemma I:

THEOREM III:—*If two pairs of opposite vertices of a quadrilateral be isogonal conjugates w. r. t. a triangle, then so will be the third pair and the pedal circles of the three pairs w. r. t. the triangle are coaxal.*

4. Let X, X′ be a pair of isogonal conjugates w. r. t. a triangle, O any point on XX′ and O′ its isogonal conjugate. Now, since the point pairs X, X′; O, O′ are both inpolar to δ all the conics of the range Γ' determined by them viz., the conics touching XX′ at O and touching O′X, O′ X′ will also be inpolar to δ. Hence, by theorem II there is a range Γ of conics confocal with those of Γ' and inscribed in the triangle. Further there is a unique conic σ belonging to both the ranges which touches XX′ at O. Now, since σ is a self corresponding member in the two ranges, the mutual orthoptic circle of σ with itself is its director circle. Therefore, by theorems I and III the director circle of σ and the pedal circles of O and X are coaxal. We have, thus, Mr. S. Narayanan's result*:

If an axis of a conic inscribed in a triangle passes through a given point O, the auxiliary circle of the conic corresponding to this axis, the pedal circle of O w. r. t. the triangle and the director circle of the inconic touching the axis at O are coaxal.

In the special case when O is the circumcentre of the triangle, the pedal circle of O is the nine points circle. By Lemma II, the director circle of the conic σ in the previous case touches the nine points circle. Hence, the pedal circle of P, P′ also touches the nine points circle of the triangle. This is the well-known extension of Feuerbach's theorem** by M′ Cay, viz.,

If an axis of an inconic of a triangle passes through the circumcentre, the corresponding auxiliary circle of the conic touches the nine points circle.

Let $\sum'$ be a conic of Γ' touching PP′ at O and $\sum$ the corresponding inscribed confocal in Γ. By Theorems I and III, the mutual orthoptic circle of $\sum$. $\sum'$, the director circle of σ and the pedal circle of O are coaxal. If, however, O be the circumcentre we know by Lemma II that the director circle of σ touches the nine points circle. Therefore, the mutual orthoptic circle of $\sum$, $\sum'$ touches the nine points circle. Thus we have proved Mr. Ramaswamy Aiyar's Theorem† viz.,

If a conic be inscribed in a triangle and a confocal to it pass through the circumcentre, the mutual orthoptic circle of the two conics touches the nine points circle.

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* Vide *An Extension of Feuerbach's Theorem* J. I. M. S. Vol. X, p. 419.

** Casey *loc cit* p. 329.

† *Proc. Edin. Math. Soc.* Vol. XV, pp. 74-75.

In the particular case when the confocal consists of a pair of foci of the inconic we get M'Cay's Theorem.

5. We now pass on to apply theorems I and III to the theory of orthopolar circles of the triangle.

Let l be any line in the plane of a triangle ABC; D, E, F the mid-points of the sides and A′, B′, C′ the projections on l from A, B, C. Now, there is a circle with centre D passing through B′ C′; another with centre E passing through C′, A′ and a third with centre F passing through A′, B′. These three circles in general have a unique circle L which cuts them all orthogonally. This circle has been called the orthopolar circle of the line l w. r. t. the triangle ABC. It can be shown* that:

LEMMA III:—*If $\sum$ be a conic inscribed in a triangle and $\sum'$ the unique confocal to $\sum$ touching l the mutual orthoptic circle of $\sum$, $\sum'$ cuts orthogonally the orthopolar circle of l w.r.t. the triangle.*

As a particular case, when $\sum'$ consists of a pair of foci of $\sum$ we have

LEMOYNE'S THEOREM:—*The pedal circle of any point on the line l w.r.t. the triangle cuts orthogonally the orthopolar circle of l.*

Let X, X′; Y, Y′; Z, Z′ be the three pairs of opposite vertices of an isogonal quadrilateral (of a triangle) whose sides are p, q, r, s and P, Q, R, S their orthopolar circles. Further, let X, Y, Z lie on p, Y, Z′, X′ on q, X, Z′, Y′ on r and X′, Y′, Z on s. By lemma IV, the pedal circles of X, Y, Z cut orthogonally the orthopolar circle P, the pedal circles of Y, Z′, X′ cut orthogonally the circle Q, etc. But, since a pair of isogonal conjugates determine the same pedal circle, the circles P, Q, R, S all cut orthogonally the pedal circles of X, Y, Z. By theorem III these pedal circles are co-axal. It follows, therefore, that the circles P, Q, R, S are also coaxal. We have thus:

THEOREM IV:—*The orthopolar circles of the sides of an isogonal quadrilateral w. r. t. the triangle belong to a coaxal system.*

Consider the range Γ′ of conics inscribed in an isogonal quadrilateral whose sides are p, q, r, s. By theorem II there is an associated range Γ of confocal conics inscribed in the triangle,

* *Vide* The Orthopolar Theory. By V. Ramaswamy Aiyar. *J. I. M. S.* Vol. IV pp. 213-215.

which will have besides its sides another line m, for a common tangent. Let m meet the sides of the triangle in A_1, B_1, C_1.

Now, the pair of points A, A_1 together constitute a conic Σ inscribed in the triangle; there is, therefore, a corresponding confocal conic Σ' in Γ'. Hence A, A_1 will be a pair of foci of Σ'. By theorems I and III their mutual orthoptic circle i.e., the auxiliary circle of Σ' corresponding to this pair of foci will belong to the coaxal system determined by the pedal circles of X, Y, Z. But, the auxiliary circle of Σ' is the circle through the feet of the perpendiculars from A (and A_1) on the tangents to Σ'. Thus, the projections from A on the lines p, q, r, s all lie on a circle coaxal with the pedal circles of X, Y, Z. Similar results hold good with regard to B and C. We are thus led to conclude that:

THEOREM V:—*The projections of each vertex of a triangle on the sides of an isogonal quadrilateral lie on a circle, and the three circles corresponding to the vertices belong to a coaxal system.*

THE MIQUEL-CLIFFORD CONFIGURATION

BY

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1. The main object of this paper is to prove two theorems, the one defining the cyclic or the bicircular quartic as a certain locus and the other revealing certain symmetric properties of the Miquel-Clifford configuration.

2. It is well known that

the locus of a point P which is such that the feet of the perpendiculars from P on four straight lines l_i in a plane lie on a circle is a cubic curve Γ passing through the 8 points of the Miquel-Clifford configuration generated by the four lines l_i. ... (2·1)

If the cubic curve Γ be regarded as lying in the projective plane, it passes through the two circular points at infinity and is then familiarly known as the circular cubic. If P be a point on Γ, the feet of the perpendiculars from P on the lines l_i lie on a circle. Hence the reflections of P about the four lines l_i lie also on a circle. Now the four lines l_i are, from the standpoint of inversive

geometry, four circles passing through the ideal point at infinity Ω of the inversive plane. Since the inverse of Γ is a circular cubic or a cyclic according as the centre of inversion does or does not lie on Γ, we have by inverting the theorem (2·1)

The locus of a point P which is such that the inverse of P w. r. t. four concurrent circles lie on a circle is a cyclic curve Γ' passing through the 8 points of the Miquel-Clifford configuration generated by the four concurrent circles. ... (2·2)

It is readily seen that Γ' passes through the limiting points of the six coaxal systems of circles defined by pairs of the given set of circles. The cyclic Γ' when regarded as lying in the projective plane is a circular cubic or a bicircular quartic according as the centres of the four given circles do or do not lie on a circle. In the inversive plane Γ' passes through Ω in the former case, while in the latter it does not.

3. The configuration generated by four lines in a plane is such that the centres of the four circles concurrent at any of its points lie on a circle—the centre circle of the four circles. The centre circle is a proper circle only in the case of the four circles passing through the Miquel point of the four lines, and it reduces to the point at ∞, Ω in the case of the four circles through Ω i.e. the four lines generating the configuration. At every other point of the configuration two of the four circles passing through the point are proper circles while the remaining two are the line circles through the point. Hence the centres of the circles through such a point lie on a line circle i.e. the line joining the centres of the two proper circles through the point. Hence by theorem (2·2) the locus Γ' associated with any tetrad of concurrent circles of the configuration is a cubic passing through the 8 points of the configuration, and when it is regarded as lying in the projective plane it passes through the circular points at infinity. Thus the curve Γ' is the same with whatever concurrent tetrad of the configuration we may start. Hence, inverting we have the theorem, that if the inverses of a point P w.r.t. a concurrent tetrad of circles of a Miquel-Clifford configuration lie on a circle then P is in the same relation with any other concurrent tetrad of circles of the configuration. Taking now a fifth circle through a point C of multiple incidence of the above configuration, if the inverses of a point P w.r.t. the five circles through C lie on a circle, it is easy to see by repeated application of the previous theorem, that the relation of P with any other concurrent

set of 5 circles of the configuration generated by the five given circles is the same; and generally we shall have the theorem:—

If the inverses of a point P w.r.t. a set of n concurrent circles of a Miquel-Clifford configuration consisting of 2^{n-1} circles and 2^{n-1} points lie on a circle, then the relation of P with any other concurrent set of n circles of the configuration is the same. ... (3·1)

The particular case of the preceding theorem obtained by taking P to be the point at infinity Ω is interesting and may be stated:

If the centres of a set of n concurrent circles of a Miquel-Clifford configuration consisting of 2^{n-1} circles and 2^{n-1} points lie on a circle then the centres of any other set of n concurrent circles of the configuration lie also on a circle.

In fact if the inverses of a point P w.r.t. the n circles of the configuration lie on a circle, we can obtain another Miquel-Clifford configuration from the given configuration by associating with each circle of the given configuration a point—namely the inverse of P w.r.t. that circle and with each point of the configuration a circle—namely the circle through the n points associated with the n circles concurrent at the point. That the set of 2^{n-1} points and 2^{n-1} circles thus obtained form a Miquel-Clifford configuration is a consequence of theorem (3·1).

It is well known that every Miquel-Clifford configuration consisting of 8 circles and 8 points is invariant under a Mobius involution†. Hence it follows that the cyclic Γ′ associated with the configuration is also invariant in the same Mobius involution. But it is known that every cyclic is automorphic with respect to three mutually harmonic involutions defined by the four mutually orthogonal circles of inversion of the cyclic. Hence it follows that

"The Mobius involution under which a Miquel-Clifford configuration is invariant is one of the three mutually harmonic involutions w. r. t. which the cyclic Γ′ associated with the configuration is automorphic".

† Ramamurthi, "On a cubic transformation in circle-geometry" *Jour. Annamalai University*, Vol. III, No. 2.

A PROPERTY OF THE MATHEMATICAL EXPECTATION

BY

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1. While working out a few of Chrystal's problems on the mathematical expectation of a drawing from a bag containing coins of two kinds we noticed that the manner of drawing was not specified and therefore we worked by the two different methods that were open to us. First we assumed that the coins were drawn one by one and replaced after every attempt and alternatively that they were drawn one by one but not replaced, or what comes to the same thing, that they were drawn at the same time. Both the methods led to the same result and we were tempted to investigate whether this is true in all cases. This note gives the details of our method of attack and draws attention to another surprising but very useful result.

2. Let a bag contain a coins of one kind and b of another, each of the former being r times the latter unit in value. If k coins are drawn it is required to find the mathematical expectation of the value of the drawing.

i. *Suppose the coins are drawn one by one and replaced.*

If $\dfrac{a}{a+b}=p$, and $\dfrac{b}{a+b}=q$ the expectation in terms of the second unit is easily seen to be

$$\begin{aligned} &p^k \cdot kr + {}_kC_1 \cdot p^{k-1}q\ [(k-1)\,r+1)] \\ &\qquad + {}_kC_2\ p^{k-2}q^2\ [(k-2)\,r+2] + \cdots\cdots \text{ to } k+1 \text{ terms.} \\ &= kr\,[\,p^k + {}_kC_1\ p^{k-1}q + {}_kC_2\ p^{k-2}q^2 + \cdots\cdots \text{ to } k+1 \text{ terms}] \\ &\qquad - (r-1)\ [{}_kC_1 \cdot p^{k-1}q + 2\cdot {}_kC_2\ p^{k-2}q^2 + \cdots\cdots k \text{ terms}]. \\ &\qquad = kr(p+q)^k - (r-1)\ kq\ (p+q)^{k-1} \\ &\qquad = kr - (r-1)\frac{kb}{a+b} = \frac{k(ar+b)}{a+b} \quad \cdots\cdots \quad (A) \end{aligned}$$

ii. *Suppose the coins are drawn one by one but not replaced*

(α) If $k<a$, $k<b$ the expectation will be given by

$$\frac{{}_aC_k}{{}_{(a+b)}C_k}\cdot kr + \frac{{}_aC_{k-1}\cdot {}_bC_1}{{}_{(a+b)}C_k}\Big[(k-1)r+1\Big] + \cdots \text{ to } (k+1) \text{ terms}$$

$$= \frac{kr}{{}_{(a+\ )}C_k}\left[{}_aC_k + {}_aC_{k-1}\cdot{}_bC_1 + \cdots\cdots (k+1) \text{ terms}\right]$$

$$- \frac{r-1}{{}_{(a+b)}C_k}\left[{}_aC_{k-1}\,{}_bC_1 + 2\cdot{}_aC_{k-2}\cdot{}_bC_2 + \cdots\cdots k \text{ terms}\right]$$

$$= kr - \frac{(r-1)b}{{}_{(a+b)}C_k}\left[{}_aC_{k-1} + {}_aC_{k-2}\cdot{}_{b-1}C_1 + \cdots\cdots\cdots k \text{ terms}\right]$$

$$= kr - \frac{(r-1)b}{{}_{(a+b)}C_k}\cdot{}_{(a+b-1)}C_{k-1} \text{ by Vandermonde's theorem}$$

$$= rk - (r-1)b\cdot\frac{k}{a+b} = \frac{k(ra+b)}{a+b} \cdots\cdots\cdots \quad \text{(B)}$$

(β) $\underline{a<k<b}$

Now the expectation will be

$$\frac{{}_aC_a\cdot{}_bC_{k-a}}{{}_{(a+b)}C_k}\left[ar+(k-a)\right] + \frac{{}_aC_{a-1}\cdot{}_bC_{k-a+1}}{{}_{(a+b)}C_k}\left[(a-1)r+(k-a+1)\right]$$
$$+ \cdots\cdots\cdots\cdots (a+1) \text{ terms}$$

$$= \frac{(ar+k-a)}{{}_{(a+b)}C_k}\left[{}_aC_a\cdot{}_bC_{k-a} + {}_aC_{a-1}\cdot{}_bC_{k-a+1} + \cdots\cdots \text{ to } (a+1) \text{ terms}\right]$$

$$- \frac{r-1}{{}_{(a+b)}C_k}\left[{}_aC_1\cdot{}_bC_{k-a+1} + 2\cdot{}_aC_2\,{}_bC_{k-a+2} + \cdots\cdots a \text{ terms.}\right]$$

By collecting the coefficients of x^k in the product of $(1+x)^a$ and $(1+x)^b$ we find

$${}_aC_a\cdot{}_bC_{k-a} + {}_aC_{a-1}\cdot{}_bC_{k-a+1} + \cdots\cdots \text{ to } (a+1) \text{ terms} = {}_{(a+b)}C_k,$$

Again

$$a{}C_1\cdot{}_bC_{k-a+1} + 2\cdot{}_aC_2\cdot{}_bC_{k-a+2} + \cdots\cdots\cdots \text{ to } a \text{ terms}$$
$$= a\,[{}_bC_{k-a+1} + {}_{(a-1)}C_1\cdot{}_bC_{k-a+2} + \cdots\cdots\cdots a \text{ terms}]$$
$$= a\,[{}_{a-1}C_{a-1}\cdot{}_bC_{k-a} + {}_{1+a-1}C_{a-2}\cdot{}_bC_{k-a+2} + \cdots\ a \text{ terms}]$$
$$= a\,[{}_{(a+b-1)}C_k] \text{ arguing as before.}$$

Hence the expectation in this case is

$$(ar+k-a) - \frac{(r-1)a\cdot(a+b-k)}{(a+b)}$$
$$= ar+k-a-(r-1)a + \frac{ka(r-1)}{a+b}$$
$$= \frac{k(ra+b)}{a+b} \cdots\cdots\cdots\cdots \quad (c)$$

If $a>k>b$ the result will be the same as can be seen by changing r into $\frac{1}{r}$, multiplying the result by r and then interchanging a and b.

(γ) $k>a\ k>b$

Now the expectation is

$$\frac{{}_aC_a \cdot {}_bC_{k-a}}{{}_{a+b}C_k} \cdot ar + \frac{{}_aC_{a-1} \cdot {}_bC_{k-a+1}}{{}_{(a+b)}C_k}\Big[(a-1)r+(k-a+1)\Big]$$

$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad + \frac{{}_aC_{k-b} \cdot {}_bC_b}{{}_{(a+b)}C_k}\Big[(k-b)r+b\Big]$$

$$= \frac{(ar+k-a)}{{}_{(a+b)}C_k}\Big[{}_aC_a \cdot {}_bC_{k-a} + {}_aC_{a-1} \cdot {}_bC_{k-a+1} + \quad . \quad . \quad . \quad . \quad + {}_aC_{k-b} \cdot {}_bC_b\Big]$$

$$-\frac{r-1}{(a+b)C_k}\Big[{}_aC_{a-1} \cdot {}_bC_{k-a+1} + 2 \cdot {}_aC_{a-2} \cdot {}_bC_{k-a+2} + \quad . \quad . \quad . \quad . \text{ to } (a+b-k) \text{ terms}\Big].$$

This reduces to

$$ar + k - a - \frac{(r-1)a \cdot (a+b-1C}{(a+b)C_k}$$

$$= \frac{k(ra+b)}{a+b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(D)}$$

3. The mathematical expectation is thus independent of the manner of drawing. It will also be seen that in all cases *it is equal to the average value of a coin in the bag multiplied by the number of coins drawn.* It may be remarked that the application of this proposition will save much labour that is now being wasted in the tedious process of calculating the individual probabilities for each case and then finding the weighted average.

4. *Example.*

A bag contains 3 sovereigns and 8 shillings. A man has the option first of drawing two coins at once or secondly, one coin and afterwards another provided the first be a shilling. Which had he better do?

The usual method

In the first option the expectation is

$$\frac{{}_3C_2}{{}_{11}C_2} \times 40 + \frac{{}_3C_1 \cdot {}_8C_1}{{}_{11}C_2} \times 21 + \frac{{}_8C_2}{{}_{11}C_2} \times 2 \text{ sh.}$$

$$= \frac{40 \times 3}{55} + \frac{24 \times 21}{55} + \frac{28 \cdot 2}{55} = \frac{680}{55} = 12\,\frac{4}{11} \text{ sh.}$$

In the second option it is

$$\frac{3}{11} \times 20 + \frac{8}{11} \cdot \frac{3}{10} \times 21 + \frac{8 \times 7}{11 \times 10} \times 2 = \frac{60}{11} + \frac{252}{55} + \frac{56}{55} \text{ sh.}$$

$$= \frac{608}{55} \text{ sh.} = 11\,\frac{3}{55} \text{ sh.} \quad \therefore \text{ The first is better.}$$

The New Method

The expectation in the first draw $= \left[\frac{68}{11} \times 2\right]$ sh. $= 12\frac{4}{11}$ sh.

" " second draw $= \frac{68}{11} + \frac{8}{11} \times \frac{67}{10}$ sh.

$= \frac{680+536}{110} = \frac{1216}{110}$ sh.

Thus the first is better. $= 11\frac{3}{55}$ sh.

N. B.—It could also be argued that the expectation of a draw of two coins simultaneously is the same as that of a draw of two, one by one without restriction. But as there is a restriction in the second option the expectation will be less. Hence the first is better.

This paper has been concerned only with the case of two kinds of coins and the method followed has been of the traditional type. In a paper to be published shortly the most general problem concerning any number of different kinds of coins is solved, but here the usual method fails and a distinctly new approach has had to be made.

GLEANINGS

The number, $M_{257} = 2^{257} - 1 =$

231584, 178474, 632390, 847141, 970017, 375815, 706539, 969331, 281128, 078915, 168015, 826259, 279871,

proved composite by Kraitchik and D. H. Lehmer, is interesting as being the largest 'genuine' composite number known, that is to say the largest number which is known to be composite but of which no factor has been found.

HARDY and WRIGHT: *An Introduction to the Theory of Numbers, 1938.*

But at this moment of exaltation Brouwer appears with something that looks suspiciously like a flaming sword in his strong right hand. The chase is on: Dedekind, in the role of Adam, and Cantor disguised as Eve at his side, are already eyeing the gate apprehensively under the stern regard of the uncompromising Dutchman. The postulational method for securing freedom from contradiction proposed by Hilbert will, says Brouwer, accomplish its end—produce no contradictions—*but* "nothing of mathematical value will be attained in this manner; a false theory which is not stopped by a contradiction is none the less false, just as a criminal policy unchecked by a reprimanding court is none the less criminal.

E. T. BELL: *Men of Mathematics*

ON FINITE BOOLEAN ALGEBRA*

BY

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This paper is devoted to a discussion of the known theory—namely, the representation of the finite Boolean Algebra as an algebra of Classes and as an algebra of Vectors. The ideas involved in the usual proofs of the representation (namely 'A finite Boolean Algebra can be regarded as an algebra of all sub-classes of a given class') seemed to the present writer to call for more clarification and precision. Hence, a systematic proof of the above theorem is attempted on the basis of the ideas of the isomorphism between groups and also between two Boolean Algebras. (See Sections IV, V of this paper). On more or less similar lines, the representation of a finite Boolean Algebra as a Vector algebra (mod. 2) is also discussed. The first section of this paper, consists of a brief account of the development and characteristic properties of the Boolean Algebra; while the second section pertains to the theory of Minimal Elements in a Boolean Algebra.

I. *The Boolean Algebra—Development and Properties.*

George Boole (in 1847) founded a symbolic method of treating the 'Algebra of Logic',—also called Boolean Algebra after him. According to its laws, we are able to solve problems of logic by symbolic methods as in pure mathematics. Subsequently after Boole, other logicians chiefly Pierce and Schröder developed what has been called the Classic or Boole-Schröder algebra of logic. This was primarily applied to the abstract system interpreted as the Calculus of Classes which sets forth the possible relations between classes and inferences from class-relations. Other interpretations are also possible for the Boole-Schröder algebra, e.g. in terms of propositions. Fundamentally, propositions are more important than classes and in the Principia Mathematica, the algebra of Propositions is dealt with first and then systematically the algebra of classes is deduced from what are known as propositional functions. In Logic, the algebra of

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* This paper formed part of a thesis for which the degree of M.Sc. was awarded by the University of Madras.

I take this opportunity to thank Dr. Vaidyanathaswamy for his kind guidance during the preparation of this paper.

classes has special importance. For, even in Boole's time, it was known that the conception of truth and falsehood could be formally expressed as an isomorphism between the propositional calculus and the two-element algebra of classes.

The term Boolean Algebra is a more modern one and has come to be accepted as denoting an algebra which has the same formal type as an algebra of all sub-classes of a given class. The calculus of propositions though in general features similar to a Boolean Algebra, yet is not a Boolean Algebra; for there are no unique propositions which correspond to the 'Zero' and the 'Universe' of the Boolean Algebra.

A Boolean Algebra can be roughly defined as a set of elements $(a, b, c, \ldots\ldots)$ with two binary operations $+$ and $\times$, and a unary operation called negation satisfying the following properties:

[Various systems of postulates for the class-calculus have been proposed by E. V. Huntington (5), B.A. Bernstein (1, 2), M. H. Stone (11) etc., with discussions as to their independence, consistency and so on. The earliest of these (which is also according to Eaton (3), Henle and Chapman (4) the best of all) is due to E. V. Huntington* (5). However, since in the present paper we are not concerned with the axiomatic point of view, we shall set down, the important properties of the Boolean Algebra without regard to the question of their independence, consistency and so on.]

(i) $a+b$, $a.b$, a' are also elements of the Boolean Algebra.

(ii) Associative and Commutative property of Addition.

$$a+b+c=(a+b)+c=a+(b+c)\,;\ a+b=b+a.$$

(iii) Associative and Commutative property of Multiplication.

$$a\cdot b\cdot c=(a\cdot b)\cdot c=a\cdot(b\cdot c)\,;\ a\cdot b=b\cdot a.$$

(iv) Double Distributive Law for Addition and Multiplication.

$$a(b+c)=a\cdot b+a\cdot c\,;\ (a+b).(a+c)=a+b\cdot c.$$

(v) Tautology Laws.

$$a+a=a\,;\ a\cdot a=a.$$

(vi) Existence of the unique elements Zero (0) and Universe (I) with the properties: $a+0=a$; $a\cdot 0=0$; $a+I=I$, $a\cdot I=a$.

(vii) Law of Absorption.

$$a+a\cdot b=a.$$

* The system referred to here, is the first of the three sets given.

(viii) The negative a' (of a) has the property that, to each a there exists a unique a' such that $a+a'=I$; $a \cdot a'=0$

(ix) Laws of Duality.

$a+b=(a' \cdot b')'$; $a \cdot b=(a'+b')'$. If $a=b$, then $(a')'=a''=a$.

Definition 1. An element a is said to 'contain' an element b ($a>b$ in symbols) if an x exists such that $a=b+x$; b is said to be 'contained' in a ($b<a$ in symbols).

An immediate consequence of this definition is that

If $a>b$ and $b>c$, then $a>c$.

(for, $a=b+x_1=c+x_2+x_1$; *i.e.* $a>c$) The following well-known theorems used in the course of this paper, are derived from the above definition and properties.

THEOREM 1. (*a*) *$a+b$ is the unique element containing a and b such that it is contained in any element containing a and b.*

(*b*) *$a \cdot b$ is the unique element contained in a and b, such that it contains any element contained in a and b.*

(In particular, it follows that if $D>a$, $D>b$, then $D>a+b$; and if $D<a$, $D<b$, then $D<a.b$).

Proof: (*a*) Let D be an element such that $D>a$, $D>b$. Then by definition 1, $D=D+D=a+x_1+b+x_2=a+b+x_1+x_2$. Thus $D>a+b$, i.e. $a+b$ is contained in an element D which contains a and b, and obviously, $a+b>a$, $a+b>b$.

(*b*) Similarly, let D be an element such that $D<a$ $D<b$. Then by definition 1, $a=D+x_1$, $b=D+x_2$. Hence, $a.b=(D+x_1).(D+x_2)=D.D+D.x_2+D.x_1+x_1.x_2$ (*by* v) $=D+x_1.x_2$ (from the properties (i)–(vii) : i.e. $D<a \cdot b$.) Thus, $a \cdot b$ contains an element D contained in a and b and obviously, $a \cdot b<a$, $a \cdot b<b$.

It is easy to show that elements with these properties are evidently unique.

THEOREM 2. *$I>a>0$.*

(Since by (vi) $a+0=a$, for all a, $a>0$. And since

$a+I=I$, $I>a$. Hence, $I>a>0$.)

THEOREM 3. *If $a>b$, then $a=b+a$; $a \cdot b=b$.*

(For, since $a>b$, by definition 1, $a=b+x$: therefore, $a \cdot b=b(b+x)$ $=b+b \cdot x=b$. Again since $a>b$, $a=b+x$.

Therefore, $a+b=b+b+x=b+x=a$.)

THEOREM 4. *If a and b be any two elements then* $a \cdot b < a$.

(This follows from the law of absorption (vii).)

THEOREM 5. *If* $a > b$, *an unique* x *can be found such that the two equations* $a=b+x$; $b \cdot x = 0$ *are satisfied.*

Proof: Since $a > b$, (by theorem 3), $a=b+a=(b+a).I$ $=(b+a)\cdot(b+b')=b.b+b.b'+a.b+a.b'=b+a.b'$ (from the properties of the Boolean Algebra, given above). And $b \cdot ab' = a \cdot bb' = 0$. Hence, there is an x namely $a \cdot b'$ to satisfy the two given equations.

Next to prove the uniqueness of x, assume that if possible x, y are two solutions of the given equations. Then we have

$$b+x=b+y \text{ (since each} = a)$$

$$b.x.=b.y \text{ (since each} = 0)$$

Multiplying the first equation by b' and adding the second equation to it, we have $x(b+b')=y(b+b')$; i.e. $x=y$. Hence, $a \cdot b'$ is unique.

THEOREM 6. *If* $a+b=0$, *then* $a=0$, $b=0$.

[for, $0=a \cdot 0 = a(a+b)=a+a \cdot b = a$. Similarly $b=0$.]

THEOREM 7. *If* $a > b$, *then* $a' < b'$.

(This follows from definition 1, (ix), and theorem 3.)

In this paper, we are concerned with a Boolean Algebra of a finite number of elements.

A simple concrete illustration of a finite Boolean Algebra, is the algebra of all sub-classes of a finite class (say F) which arises when the sum and product of two sub-classes (a) and (b) are defined as the sub-classes resulting from the 'union' and the 'intersection' of (a) and (b) respectively. The Null-class and the Finite class (F) are the Zero (O) and the Universe (I) of this Boolean Algebra. The negative (a') of a class (a) is the class of all elements in I which do not belong to (a). All the properties given above for the Boolean Algebra can be directly verified for this algebra of sub-classes. It is clear that a sub-class (a) 'contains' a sub-class (b) according to the definition given above, if (b) is part of (a).

II. *The Minimals* of a Finite Boolean Algebra.*

The important idea of Minimals in a Boolean Algebra was first introduced by E. V. Huntington (5) under the name of Irreducible Elements. The fundamental properties of minimals are two:—

* See B.A. Bernstein (1), for properties and alternative definition of minimals.

namely any element of the Boolean Algebra can be expressed as a sum of minimals uniquely (see theorem 12) and the product of two distinct minimals is Zero (see theorem 10). Later on, B.A. Bernstein (1) developed the theory of minimals and their duals (which he calls Maximals) more systematically. Obviously Maximals have the same relation to the operation $\times$ as minimals have to the operation $+$.

The conception of minimals is important in many ways. For instance, the theory of minimals and maximals is the counterpart in the Boolean Algebra, of the known theory in the propositional calculus of the Disjunctive and Conjunctive normal forms of a general logical expression (logische ausdrücke) involving a given number of propositions (A_1, A_2,........., A_n) as dealt with by Hilbert and Ackermann (7). Again, the theory of minimals is the key to the proof of the theorem* we have set out to prove in this paper (i.e. a finite Boolean Algebra can be regarded as an algebra of all sub-classes of a finite class). Similar properties true for the generalised Boolean Algebra have been proved by M. H. Stone (10) who has for the purpose generalised the theory of minimals (or atomic elements as he calls them). Below, I give a definition** of minimals and some of their characteristic properties.

Definition 2. A non-zero element of a finite Boolean Algebra B is said to be a minimal if it does not contain any element other than itself and zero.

THEOREM 8. *Every non-zero element of a finite Boolean Algebra B contains a minimal.*

Proof: Let a be an arbitrary element ($\neq 0$) of B. Then since $a \neq 0$, a is either a minimal or not a minimal. In the former case, since a contains itself, the theorem is once proved. In the latter case, it follows from definition 2, that a contains an element b different from itself and from zero. Again b can be a minimal or not a minimal. In the former case the theorem is proved since a contains a minimal b. In the latter case, it follows in the same way as before that there is an element c contained in b different from b and different from zero. Proceeding thus, since the total number of elements contained in a is finite (because B is finite) we arrive at a minimal contained in a in a finite number of steps.

* An investigation which establishes this theorem for the general Boolean Algebra, has been made by M. H. Stone (10).

** It is easy to show that this definition is identical with Bernstein's definition. See my paper 'On Symmetric Functions of n elements in a Boolean Algebra.'—*Journal of Ind. Math. Soc.* p. 4., Vol. II, No. 5.

THEOREM 9. *If a be an arbitrary element, and m be any minimal, then either $a.m=0$ or $a.m=m$.*

Proof: For by theorem 4, $am<m$. But since m is a minimal, the only elements which are $<m$ are 0 and m. Hence, $am=0$ or $am=m$.

THEOREM 10. *The product of two distinct minimals is zero.*

For, if m_1, m_2, be two different minimals, by theorem 9 above, $m_1 \cdot m_2=0$ or m_1; $m_2 \cdot m_1=0$ or m_2. But since $m_1 \neq m_2$, the two statements can be true only if $m_1 m_2=0$.

THEOREM 11. *The only element which is orthogonal * to all minimals is zero.*

Proof: Let $xm=0$ for every minimal m. We shall show this is impossible unless x is zero. For, if $x \neq 0$, by theorem 9 it contains a minimal m so that $x \cdot m=0$. This contradicts our hypothesis namely that $x \cdot m=0$, for every minimal m. Hence, $x=0$.

THEOREM 12. *Every element can be expressed uniquely as a sum of minimals.*

Proof: Let a be an arbitrary element $(\neq 0)$, and let all the minimals of the finite Boolean Algebra B be $m_1, m_2, \cdots m_k, m_{k+1}, \cdots \cdots m_r$. If m be any one of the minimals, then from theorem 9, $a.m=0$ or $a.m=m$. Hence, all the minimals can be divided into two classes such that for the minimals of the first class $a.m=m$, and for the minimals of the second class $a.m=0$; i.e. the minimals of the first class are contained in a and the minimals of the second class are not contained in a. And since $a \neq 0$, there always exist by theorem 8 minimals which are contained in a. Let $m_1, \cdots\cdots, m_k (k \geqslant 1)$ be the minimals contained in a, and $m_{k+1}, m_{k+2}, \cdots\cdots$ m_r be those not contained in a. Then since $a>m_1$, $a>m_2$, $\cdots\cdots$, $a>m_k$, we have by Theorem 1 $a>m_1+m_2+\cdots\cdots+m_k$. Hence by Theorem 5, we can always find an unique x such that

$$a=(m_1+m_2+\cdots\cdots+m_k)+x.$$

$$\text{and, } x(m_1+m_2+\cdots\cdots+m_k)=0.$$

And by (iii) of Sec. I and Theorem 6, it follows from the second equation, $m_1 x=0$, $m_2 x=0$, $\cdots\cdots$, $m_k x=0$.

Next considering the minimals $(m_{k+1}, m_{k+2}, \cdots\cdots m_r)$ of the second class, since by Theorem 9 $a.m_{k+1}=0$; $\cdots$; $a.m_k=0$, and also

* In a Boolean Algebra a is said to be orthogonal to b, if $a.b=o$.—M. H. Stone (10).

since $a > x$, it follows from Definition 1, (iii) of Sec. I and Theorem 6 that

$$x.m_{k+1} = 0, \quad x.m_{k+2} = 0, \cdots\cdots\cdots, \quad x.m_k = 0.$$

Thus we see, $x.m_1 = 0$, $x.m_2 = 0$, $\cdots\cdots$ $x.m_r = 0$.

Hence by Theorem 4, $x = 0$.

Therefore, $a = m_1 + m_2 + \cdots\cdots + m_k$ = the sum of the minimals contained in a; and this expression is evidently unique. For the sake of convenience, we shall refer to these minimals (i. e. the minimals contained in a) as 'minimals of a'.

THEOREM 13. *The minimals of an algebra K of all sub-classes of a finite class C, are the one-element sub-classes of C.*

For, in the algebra of all sub-classes of a finite class C, the minimals by definition 2, are those sub-classes which do not contain any sub-classes other than themselves and the Null-class. Such sub-classes are obviously the one-element sub-classes of C.

III. *The Abelian Disjunction* group constituted by the Elements of B.*

Contrary to what happens in ordinary algebra, the elements of a Boolean Algebra B do not form a group with respect to either of the operations + and ×. But however, we can define an operation called disjunction and denoted by ⊕ for which the elements form a group which is abelian.

Definition 3. The disjoint of two elements a, b is given by a $a \oplus b = ab' + a'b$.

[Hence, in the algebra of classes, the disjoint of two classes (a) (b) is defined as that class of elements which belong to one but not to both of the classes (a) and (b)].

Some chief properties † of disjunction are:

(a) It is commutative and associative.

(b) $a \oplus 0 = a$; $a \oplus a = 0$; $a \oplus \mathrm{I} = a'$.

(c) It distributes x.

* See B. A. Bernstein (2); See also M. H. Stone (11), and also p. 103-105. Proc. of the Nat. Acad. of Sciences 1935. But the dual operation namely conjunction, is not studied by M. H. Stone in the above two papers. See also R. Vaidyanathaswamy (12).

† For proof of the property (c) see my paper—"On Symmetric Functions etc." loc. cit. page 8, th. 2.

தமிழ்நாடு ஆவணக் காப்பக

(*d*) $a+b=(a\oplus b)+a.b.$

(for, from definition 3, $a\oplus b+a.b=a.b'+a'.b+a.b.$

$=a(b+b')+a'b=a+b$).

Corollary. If $ab=0$, $a+b=a\oplus b$.

From (*b*) we can at once see that the identity element of the group with respect to disjunction is 0, and that each element has itself as its own inverse. Also since, $a\oplus a=0$, each element except of course 0, is of order 2. Further, since disjunction is commutative, the group is abelian.

Now from the theory of finite abelian groups, we know that any such group possesses a set of basis-elements $i_1, i_2, \ldots i_n$; so that any element of the group can be uniquely expressed in terms of the elements i, by means of the group operation. Hence, for the finite abelian group constituted by the elements of the finite Boolean Algebra B with disjunction as group operation, there exists a basis (say) $i_1, i_2, \ldots\ldots i_n$, which we may call a **Disjunction-Basis** of B; and any element of the disjunction-group B, can be expressed as the continued disjoint of elements chosen from these i's each i occurring possibly more than once. But since every element, and therefore every basis-element is of order 2 with respect to disjunction, it follows that each i occurs only once or not at all in the simplified expression of a as the continued disjoint of the i's. Thus, $i_1, i_2, \ldots\ldots i_n$, being a disjunction-basis of the Boolean Algebra, any element a can be expressed as the continued disjoint of a unique sub-set of these i's.

It is clear that a disjunction-basis of B is not unique. We shall now proceed to show that the minimals of B constitute a special disjunction-basis.

THEOREM 14. *The set of minimals of a finite Boolean Algebra* (*B*) *is a disjunction-basis of B.*

Proof; For, from the corollary of (*d*), it follows that the sum of a finite number of elements is equal to their continued disjoint provided the product of every two of the elements is zero. Hence, by theorem (10) the sum of a number of minimals is equivalent to their continued disjoint. And since by theorem 12 any element of B can be expressed uniquely as the sum of minimals of B, it follows that any element can be expressed uniquely as the continued disjoint of minimals of B. Thus the minimals of B constitute a Disjunction-Basis.

THEOREM 15. *The minimals of $a \cdot b$, are the minimals which belong to both a and b.*

Proof: Let i_p, i_q, . . . , i_x, i_y, . . . be the minimals of a, b, respectively. Then by Theorem 14 we have,

$$a = i_p \oplus i_q \oplus i_r \oplus \ldots\ldots$$
$$b = i_x \oplus i_y \oplus i_z \oplus \ldots\ldots$$

Then, $a \cdot b = (i_p \oplus i_q \oplus i_r \oplus \ \ldots) \cdot (i_x \oplus i_y \oplus i_z \ \ldots)$
= the continued disjoint of those minimals which belong to both a and b (since $i_p \times i_q = 0$)

IV. *Isomorphism between two Boolean Algebras.*

Definition 4. Two Boolean Algebras L, L_1 are said to be isomorphic if there is a one-to-one reversible correspondence between the elements of L, L_1 which preserves the 'containing' relation (i.e. if a, b; a_1, b_1; be pairs of corresponding elements of L, L_1, if $a > b$, then $a_1 > b_1$ and *vice-versa.*

The following theorems regarding the isomorphism of two Boolean Algebras L, L_1, are fundamental.

THEOREM 16. *If two Boolean Algebras are isomorphic, then*

(i) *O, I of either correspond to O, I of the other.*

(ii) *Minimals of one correspond to minimals of the other.*

(iii) *Sum and product of elements of one algebra, correspond to the sum and product of the corresponding elements of the other algebra.*

(iv) *The negative of an element of one algebra corresponds to the negative of the corresponding element of the other.*

(v) *L, L_1, considered as disjunction-groups are also isomorphic.*

Proofs:—(i) Let a, a_1 be a set of corresponding elements of L, L_1 and suppose x, y, of L_1 correspond to 0, I of L. Then since by Theorem 2, $I > a > 0$ for every a, it follows from the definition of isomorphism that $y > a_1 > x$ for every a_1. This can be true only if $x = 0$ and $y = I$.

(ii) Let m be a minimal of L and let an element m_1 of L_1 correspond to m. Suppose if possible m_1 is not a minimal. Then from definition 2, m_1 must contain an x_1 such that $x_1 \neq m_1$ *and* $\neq 0$. Now by (i), 0 of L corresponds to 0 of L_1; let x of L correspond to

x_1 of l_1. Then from definition 4, it follows that $m > x$ such that $x \neq m$, $x \neq 0$. This contradicts our hypothesis namely that m is a minimal of L. Hence m_1 must be a minimal of L_1.

(iii) Let a, b; a_1, b_1, be two pairs of corresponding elements of L, L_1 and let x of L_1 correspond to $a+b$ of L. We have to prove $x = a_1 + b_1$. Now by Theorem I, $a+b$ is the unique element $>a$ and $>b$ and contained in any element which is $>a$ and $>b$. Therefore, since L, L_1 are isomorphic, x is the unique element contained in any element which is $>a_1$ and $>b_1$ and which is itself $>a_1$ and $>b_1$ i.e. by Theorem I, $x = a_1 + b_1$. Thus, $a_1 + b_1$ corresponds to $a+b$. Similarly we can prove that $a_1 \cdot b_1$ corresponds to ab.

(iv) Let a, a_1 be a pair of corresponding elements of L, L_1. Also let a' be the negative of a. Let us assume x of L_1 to correspond to a'. Now from (vii) of Sec. I. the equations $aa' = 0$, $a + a' = I$ hold for every a and its unique negative a'. Now since L, L_1 are isomorphic, we have from (iii) above, $a_1 \cdot x$, $a_1 + x$ corresponding to aa', $a + a'$ respectively. And since 0, I of L_1 correspond to 0, I of L, the equations $a_1 \cdot x = 0$, $a_1 + x = I$, ($I > a$ by Theorem 2) hold for a_1, x and by Theorem 5 have a unique solution for x namely $a'_1 \cdot I = a'_1$ by (v) of Sec. I. Hence, a'_1 (the unique negative of a_1) corresponds to a'.

(v) Let a, b; a_1, b_1 be two pairs of corresponding elements of L, L_1. By (iv) a'_1, b'_1 correspond to a', b' respectively. Hence from (iii) $a_1 \cdot b'_1$ corresponds to $a \cdot b'$, and $a'_1 \cdot b_1$ to $a' \cdot b$; Therefore, $a_1 \cdot b'_1 + a'_1 \cdot b_1$ corresponds to $a \cdot b' + a' \cdot b$ i.e. $a_1 \oplus b_1$ corresponds to $a \oplus b$. Thus the two disjunction-groups constituted by L, L_1 are isomorphic.

THEOREM 17: *A reversible one-to-one correspondence between two Boolean Algebras L, L_1 is a Boolean isomorphism if either*

(i) *The product of any two elements of one algebra corresponds to the product of the corresponding elements of the other,*

Or, (ii) *The sum of any two elements of one, corresponds to the sum of the corresponding elements of the other.*

Proof:—(i) Let a, b; a_1, b_1 be two pairs of corresponding elements of L, L_1 and let $a > b$. Then by Theorem 3, $ab = b$. Since by hypothesis $a_1 \cdot b_1$ corresponds to ab, we have $a_1 \cdot b_1 = b_1$. i.e. $a_1 > b_1$. Thus by definition 4, L, L_1 are isomorphic.

(ii) Similarly for the second case, if $a > b$, $a + b = a$. And since $a_1 + b_1$ corresponds by hypothesis to $a + b$, we have $a_1 + b_1 = a_1$, i.e. $a_1 > b_1$. Thus by definition 4, L, L_1 are isomorphic.

We may remark in connection with Theorem 16 (v) that the isomorphism of L, L_1 as disjunction-groups does not imply their isomorphism as Boolean Algebras. The condition for this is given in the next theorem.

THEOREM 18. *If a group isomorphism* Γ, *can be established between the disjunction-groups constituted by two Boolean Algebras L,* L_1 *the necessary and sufficient condition for* Γ *to be a Boolean isomorphism, is that the minimals of the algebras correspond in* Γ.

Proof: Theorem 16 (i) and (iv) prove the necessity of the condition. To prove the sufficiency, let us assume the minimals $i'_p, i'_q, i'_r, \cdots\cdots i'_x, i'_y, i'_z, \cdots\cdots$ of L_1 correspond to the minimals $i_p, i_q, i_r, \cdots\cdots i_x, i_y, i_z, \cdots\cdots$ of L in the group of isomorphism Γ. Also let a, b; a_1, b_1 be two pairs of corresponding elements of L, L_1. Now from Theorem 14 any element of a finite Boolean Algebra can be expressed as the continued disjoint of the minimals of the algebras; and by the same theorem, these minimals constitute a disjunction-basis of the algebra.

$$\text{Thus } a = i_p \oplus i_q \oplus i_r \oplus \cdots\cdots\cdots$$
$$b = i_x \oplus i_y \oplus i_{z'} \oplus \cdots\cdots\cdots$$

And from hypothesis it is clear that the minimals of a_1, b_1 correspond to the minimals of a, b respectively. Hence the minimals common to a_1, b_1 correspond to the minimals common to a, b (by theorem 15.) Therefore it follows that $a_1 \cdot b_1$ (= the continued disjoint of the minimals common to a_1, b_1 by theorem 15.) corresponds to $a \cdot b$ (= the continued disjoint of the minimals common to a, b by the same theorem). Thus by Theorem 17, the two algebras are isomorphic.

V. *The correspondence between the Finite Boolean Algebra B and the Classes of its Minimals.*

Let $i_1, \cdots\cdots i_n$ be a disjunction-basis of B. Then by expressing a as the disjoint of the i's (say $a = i_p \oplus i_q \oplus i_r \oplus \cdots\cdots$) we see that an element of the Boolean Algebra B corresponds to a definite sub-set $i_p, i_q, i_r, \cdots\cdots$) of the i's. Conversely, to each sub-set ($i_p, i_q, i_r, \cdots\cdots$) of the disjunction-basis, there corresponds the unique element defined by $a = i_p \oplus i_q \oplus i_r \oplus \cdots\cdots$ Thus we have established a one-to-one reversible correspondence between the elements of B, and the sub-sets of the i's which are the elements of the algebra K of all the sub-classes of the disjunction-basis. We shall now prove that *this one-to-one reversible correspondence between the Boolean Algebras B, K, is an isomorphism between the disjunction-groups B, K.*

Proof: Let $a = i_p \oplus i_q \oplus i_r \oplus \cdots\cdots\cdots$.
$b = i_x \oplus i_y \oplus i_z \oplus \cdots\cdots\cdots$

so that the elements a, b of the boolean algebra B correspond respectively to the sub-classes $(i_p, i_q, i_r \cdots\cdot)$, $(i_x, i_y, i_z \cdots)$ (i.e. elements) of the algebra K of all the sub-classes of the disjunction-basis of B. Then, $a \oplus b = (i_p \oplus i_q \oplus i_r \oplus \cdots) \oplus (i_x \oplus i_y \oplus i_z \oplus \cdots)$. Since $i_k \oplus i_k = 0$, the above expression for $a \oplus b$ reduces to the continued disjoint of the minimals which belong to one but not to both of a and b i.e. to the minimals which belong to the disjoint-class of the sub-classes $(i_p, i_q, i_r, \cdots\cdot)$, $(i_x, i_y, i_z, \cdots\cdot)$. Thus $a \oplus b$ corresponds to that sub-class of K which is the disjoint of the sub-classes $(i_p, i_q, i_r, \cdots\cdot)$, $(i_x, i_y, i_z, \cdots\cdot)$; i.e. the disjoint of two elements of B corresponds to the disjoint of the corresponding elements of K. Hence the result.

It is clear that in this group isomorphism, any element i_k of the disjunction-basis $i_1, i_2, \cdots\cdot, i_n$, corresponds to the one-element sub-class (i_k) and *vice versa*. But as already remarked above, this correspondence will not be in general a Boolean isomorphism. The necessary and sufficient condition for this is by theorem 18, that the minimals of B, correspond to the minimals of K. Now by theorem 13 the minimals of K; are the one-element sub-classes $(i_1), (i_2) \cdots\cdot (i_k)$ of the disjunction-basis i. Hence the elements $i_1, i_2, \cdots, i_n$ of B which correspond to their one-element sub-classes $(i_1), (i_2), \cdots, (i_n)$ must be minimals of B.

Thus the necessary and sufficient condition for the correspondence between the elements of B and the sub-classes of a disjunction-basis to be a Boolean isomorphism, is that the elements $i_1, i_2, \cdots\cdot, i_n$ be the minimals of B. Hence, by choosing the disjunction-basis as the set of minimals of B, B and K are isomorphic; hence *any finite Boolean Algebra can be represented as the algebra of all sub-classes of a finite class*,—namely the disjunction-basis constituted by the minimals of the Boolean Algebra.

Now, if the disjunction-basis contains n elements, the total number of sub-classes of the disjunction-basis is 2^n. Hence, in particular, we can conclude that the order of a finite Boolean Algebra with n minimals is 2^n.

VI. *Vector Representation of the Boolean Algebra.*

*It is well known that an abelian group is the direct product of the cyclical groups. And in particular, the disjunction-group of order 2^n of the Boolean Algebra, whose n basis elements are all of

* Speiser (9).

order 2 can be represented as the direct product of n cyclical groups of order 2. These cyclical groups are abstractly identical with the additive groups of residue classes (mod. 2.). Hence, the disjunction-group of the Boolean Algebra is abstractly identical with the additive group of vector's $(\varepsilon_1, \varepsilon_2 \ldots \varepsilon_n)$ where the ε's are the residue classes (mod. 2) and hence $= 0$ or 1 (mod. 2.)

We shall now obtain in a direct manner, the representation of the Boolean Algebra as a vector algebra (mod. 2.) so that the disjoint of two elements may correspond to the sum (mod. 2) of two corresponding vectors. Since we have already shown that a finite Boolean algebra can be regarded as an algebra of sub-classes of its minimals, it is sufficient to effect this representation for the algebra K of all sub-classes of a finite class of n elements $(a_1, a_2, a_3, \ldots, a_n)$. Now let us make each sub-class p of K to correspond to a vector $P \equiv (e_1, e_2, \ldots, e_n)$ where $e_i = 1$ or 0 according as the ith element a does or does not occur in p. Clearly, these vectors P form an abelian group of order 2^n with respect to addition (mod. 2) the identity-element being the vector $(0, 0, \ldots, 0)$ which corresponds to the null-class or the null-element of the Boolean Algebra. We now show that the above *one-to-one correspondence between the sub-classes and the vectors (mod. 2) is a group-isomorphism i.e. the disjunction-group of all sub-classes of the finite class of n elements and the additive group (mod. 2) of the corresponding vectors are isomorphic.* For, let P, Q correspond to p, q respectively. Then since by definition (3) of the disjoint of two elements, a_i occurs in $p \oplus q$ only if it occurs in either p or q but not in both, it is clear that the ith component of the vector corresponding to $p \oplus q$ is the sum (mod. 2) of the ith components of the vectors P, Q. Hence $p \oplus q$ corresponds to $P + Q$ and therefore, the disjunction-group of all sub-classes of a finite class of n element $(a_1, a_2, \ldots a_n)$ and the additive group of the corresponding vectors are isomorphic.

Now, it is clear that since disjunction distributes $\times$ (c of sec. III), the Boolean Algebra B forms a ring in the usual sense with respect to $\oplus$ and $\times$. If we define the product of the vectors (mod. 2) as a vector whose i^{th} component is the product of the corresponding i^{th} components of the two vectors, we can see that these vectors form a ring with respect to $+$ and $\times$. These two rings are isomorphic as will be seen from the following theorem.

THEOREM 19. *The group isomorphism between the disjunction-group of the Boolean Algebra and the additive group of vectors is also a ring isomorphism between the Boolean ring w.r.t.* $\oplus$ *and* $\times$ *and the ring of the vectors w.r.t.* $+$ *and* $\times$.

Proof: As before let P, Q correspond to p, q respectively. And let α_i, β_i, γ_i be the ith components of P, Q and the vector corresponding to $p \times q$ respectively. Then from the definition of the correspondence of the vectors, it is clear that $\alpha_i = 1$ or 0; $\beta_i = 1$ or 0; $\gamma_i = 1$ or 0 (mod. 2) according as a_i does or does not occur in the sub-classes p, q, $p \times q$ respectively. Now, $\gamma_i = 1$ (mod. 2) if and only if $\alpha_i = \beta_i = 1$, and $= 0$, in all other cases. For the vector corresponding to $p \times q$ can have 1 as the ith component only if a_i occurs in both p and q.

Hence from the definition of the product of two vectors we have $P \times Q$ corresponding to $p \times q$. Hence by Theorem 17 (i) the two rings are isomorphic.

Hence finally we have the result that a finite Boolean algebra can be exhibited as a vector algebra (mod. 2.) in such a manner that the disjoint and the product of the Boolean algebra correspond to the sum and the product (defined in the special manner indicated above), of the vector algebra.

We may observe that the Boolean Matrix Algebra considered by Mr. Narasinga Rao (8), is in substance only an algebra of vectors of n components of the type considered above: i.e. no specific notions regarding matrices as such need be involved.

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NOTES AND DISCUSSIONS

Squares in Arithmetic Progression

To solve $A^2 - B^2 = B^2 - C^2$, when we have also $A + B = D^3$, $B + C = E^3$.

Denote by x^2, y^2, z^2 three squares in arithmetical progression, then

$$x = (-m^2 + 2mn + n^2)\rho\ ;\ \ y = (m^2 + n^2)\rho\ ;\ \ z = (m^2 + 2mn - n^2)\rho.$$

We have:

$$x + y = 2n(m + n)\rho\ ;\ \ y + z = 2m(m + n)\rho\ ;$$

so that replacing m, n by $2m^3$ and $2n^3$ and taking $\rho = (m^3 + n^3)^2$ we have:

$$A = 4(-m^6 + 2m^3n^3 + n^6)(m^2 + n^3)^2$$

$$B = 4(m^6 + n^6)(m^3 + n^3)^2$$

$$C = 4(m^6 + 2m^3n^3 - n^6)(m^3 + n^3)^2$$

$$A + B = 8n^3(m^3 + n^3)^2$$

$$B + C = 8m^3(m^3 + n^3)^2.$$

Put $m = 2$, $n = 1$

$$A = -15228\ ;\ \ B = 21060\ ;\ \ C = 25596.$$

$$A^2 - B^2 = B^2 - C^2 = 211631616$$

$$A + B = 18^3\ ;\ \ B + C = 36^3.$$

Put $m = 4$, $n = 3$. Then

$$A = 2\ 948\ 036$$

$$B = 159\ 823\ 300$$

$$C = 226\ 005\ 052$$

$$A + B = 162\ 771\ 336 = 546^3$$

$$B + C = 385\ 828\ 352 = 728^3$$

$$A^2 - B^2 = B^2 - C^2 = 25\ 534\ 796\ 306\ 632\ 704.$$

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Lecture Notes

The Editor invites for publication under this heading brief criticisms of results and proofs in Text Books, Questions from University examination papers; and such other material as is likely to be helpful in the collegiate class room.

I. *On the Convergency of the series* $\sum 1/n^k$.

1. Let $S_n \equiv 1 + \frac{1}{2^k} + \frac{1}{3^k} + \cdots\cdots \frac{1}{n^k}$.

Case (i) Let $k>1$. We have now

$$S_{2n+1} = 1 + \frac{1}{3^k} + \frac{1}{5^k} + \cdots \frac{1}{(2n+1)^k} + \left(\frac{1}{2^k} + \frac{1}{4^k} + \cdots \frac{1}{(2n)^k}\right)$$

$$< 1 + \frac{1}{2^k} + \frac{1}{4^k} \cdots\cdots \frac{1}{(2n)^k} + (\cdots\cdots\cdots)$$

$$\therefore \quad S_{2n+1} < 1 + \frac{2}{2^k}\left(1 + \frac{1}{2^k} + \frac{1}{3^k} \cdots \frac{1}{n^k}\right)$$

that is $S_n < S_{2n+1} < 1 + \frac{1}{2^{k-1}} S_n$

$$\therefore \quad S_n < 2^{k-1}/(2^{k-1} - 1)$$

Thus S_n is bounded above as $n \to \infty$. Hence the series cannot diverge and, being a positive term series, it cannot oscillate. So it is convergent.

2. Case (ii) Let $k = 1$. Then

$$S_{2n} = \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} \cdots \frac{1}{(2n-1)} + \left(\frac{1}{2} + \frac{1}{4} \cdots\cdots + \frac{1}{2n}\right)$$

$$> \frac{1}{2} + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} \cdots\cdots \frac{1}{2n}\right) + (\cdots\cdots\cdots\cdots)$$

$$\therefore \quad S_{2n} > \frac{1}{2} + 2\left(\frac{1}{2} + \frac{1}{4} + \cdots \frac{1}{2n}\right) = \frac{1}{2} + S_n$$

By repeated applications the sum of 2^r terms is seen to be greater than $\frac{1}{2} r$ and hence the series diverges.

3. Case (iii) Let $k < 1$.

This is easily seen to be divergent as each term is greater than the corresponding terms of $\sum 1/n$.

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II. *To determine whether a given point is in the acute or obtuse angle between two given lines.*

The following is a simple method.

Let P be the point, O the intersection of the lines. Draw PM, PN perpendiculars to each of the lines. P lies in the acute or obtuse angle between the lines according as

$$PM^2 + PN^2 < \text{or} > MN^2$$

If the lines are taken in the form

$$L_i\ (x,\ y) \equiv p_i - x\ \cos\ \alpha_i - y\ \sin\ \alpha_i = 0 \qquad (i = 1,\ 2)$$

then the point $(x',\ y')$ lies in the acute or obtuse angle between the lines according as

$$L_1\ (x',\ y')\ L_2(x',\ y')\ \cos\ (\alpha_2 - \alpha_1) < \text{or} > 0.$$

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Prof. Hukam Chand of S. D. College, Lahore, discusses the same problem taking the general equation of the lines and obtains the condition

$$(l_1x' + m_1y' + n_1)(l_2x' + m_2y' + n_2)(l_1l_2 + m_1m_2) < \text{or} > 0.$$

For a treatment of a similar problem regarding bisectors vide: *The Math. Student* Vol. III, p. 105.

III. *Trigonometric Formulae.*

In this note the well-known Trigonometric formulae for the sum or difference of two angles are obtained from the same figure which is simpler than those usually used for the purpose.

In the triangle ABC let AD be perpendicular to BC and CE perpendicular to BA produced.

1. $BA \cdot CE = 2 \times$ area of the triangle $ABC = BC \cdot AD$
$= DC \cdot AD + BD \cdot AD.$

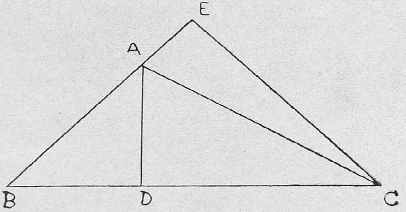

FIG. I

Dividing both sides by $BA \cdot AC$, we get

$\text{Sin}\ (B + C) = \sin\ CAE = \sin\ B \cdot \cos\ C + \cos\ B\ \sin\ C$

2. $AD^2 = AB^2 - BD^2$

$$= BA\ (BE - AE) - BD\ (BC - DC)$$

$$= BD \cdot DC - BA \cdot AE\ [\because\ BA \cdot BE = BD \cdot BC]$$

Dividing both sides by $BA \cdot AC$, we get

$$\sin B \sin C = \cos B \cdot \cos C - \cos (B + C)$$

or $\cos (B + C) = \cos B \cdot \cos C - \sin B \cdot \sin C$

3. $$\text{Sin } ACB = \frac{AD}{AC} = \frac{BC \cdot AD}{BC \cdot AC} = \frac{BA \cdot EC}{BC \cdot AC}$$

$$= \frac{BE \cdot EC - AE \cdot EC}{BC \cdot AC}$$

$$= \cos EBC \cdot \sin EAC - \cos EAC \cdot \sin EBC.$$

$\angle ACB = A - B$, if $\angle EAC = A$ and $\angle EBC = B$.

$\therefore\ \sin (A - B) = \sin A \cdot \cos B - \cos A \cdot \sin B.$

4. $$\cos ACB = \frac{CD}{AC} \cdot \frac{BC}{BC} = \frac{BC^2 - BD \cdot BC}{AC \cdot BC} = \frac{BC^2 - BA \cdot BE}{AC \cdot BC}$$

$$= \frac{EC^2 + BE^2 - BA \cdot BE}{AC \cdot BC} = \frac{EC^2 + BE \cdot AE}{AC \cdot BC}$$

$$= \sin CAE \cdot \sin CBE + \cos CAE \cdot \cos CBE$$

$\therefore\ \cos (A - B) = \sin A \cdot \sin B + \cos A \cdot \cos B.$

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IV. *On Stationary positions of planets.*

A planet will appear stationary to an observer when its representative point on his celestial sphere has no velocity; in other words when the actual velocity of the planet in space is along the line of sight either way. The determination of these stationary positions as given by Ball and Godfrey are somewhat long as they ignore the kinematical aspect of the question. Barlow and Bryan obtain equation (A) from simple geometrical considerations but have not proceeded further. The following purely kinematical treatment though straight forward, may not be out of place under "Lecture Notes"

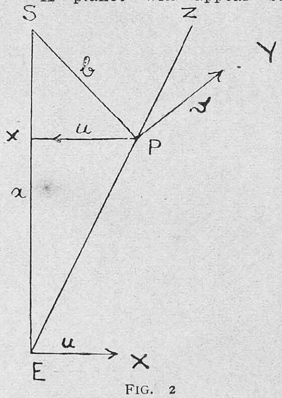

FIG. 2

Let S be the Sun, E the Earth and P a Planet. It is assumed usual, that the two orbits are as circular and coplanar.

Let 'u' be the velocity of E, 'v' that of P. Their directions are perpendicular to SE and SP respectively, and are indicated in the figure by EX and PY.

Let SE $=a$, SP $=b$, and Z a point, on EP produced.

Now we have to express the condition that the velocity of P relative to E is along EP or PE; hence the component of this relative velocity perpendicular to EP should vanish.

For this purpose, reduce E to rest by imposing on P a velocity equal and opposite to 'u', i.e., in the direction PX′ where PX′ is parallel to EX. Now P has two velocities, 'u' and 'v'. Their component perpendicular to EP is

$$u.\ \sin \mathrm{EPX}' - v.\ \sin \mathrm{YPZ}$$
$$= u.\ \sin \mathrm{PEX} - v.\ \cos \mathrm{SPZ}$$
$$= u.\ \cos \mathrm{E} + v.\ \cos \mathrm{P} \text{ (where E and P stand for } \angle\mathrm{SEP} \text{ and } \angle\mathrm{SPE.)}$$

Thus $u \cos \mathrm{E} + v \cos \mathrm{P} = 0 \quad \cdot \cdot \cdot \cdot \cdot \quad$ (A)

Now we proceed by the usual method, viz.,

$$a = b.\ \cos\theta + \mathrm{PE}.\ \cos \mathrm{E}.$$
$$b = a.\ \cos\theta + \mathrm{PE}.\ \cos.\ \mathrm{P}.$$

Hence, $$\frac{a - b.\ \cos\theta}{b - a\ \cos\theta} = \frac{\cos.\ \mathrm{E}}{\cos \mathrm{P}} = -\frac{v}{u} \text{ from (A)}$$
$$= -\frac{b^{-1/2}}{a^{-1/2}} = -\frac{\sqrt{a}}{\sqrt{b}}$$

$$\sqrt{a}\ (b - a\ \cos.\ \theta) + \sqrt{b}\ (a - b\ \cos\ \theta) = 0.$$

$$\therefore\ \cos\theta = \frac{(\sqrt{a} + \sqrt{b})\ \sqrt{(ab)}}{a\sqrt{a} + b\sqrt{b}} = \frac{\sqrt{(ab)}}{a + b - \sqrt{(ab)}}$$

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A Note on an Inequality

(Suggested by Prof. K. S. K. IYENGAR'S note in the June number of *The Mathematics Student.*)

1° If $f(x)$ is differentiable in $(a \leqslant x \leqslant b)$ and if $f(a)=f(b)=0$, then $\left| \int_a^b f(x)dx \right| < \frac{M(b-a)^2}{4}$, where $|f'(x)| \leqslant M$ in $a \leqslant x \leqslant b$.

Proof:—Geometrically this is almost obvious.

Let A, B be the points $(a, 0)$ and $(b, 0)$ and let K be the point $\frac{1}{2}(a+b)$, $\frac{1}{2}(b-a)M$ so that KAB is an isosceles triangle with $K\hat{A}B = K\hat{B}A = \tan^{-1} M$. Under the given conditions the curve $AB\{y=f(x)\}$ must lie *within* the triangle.

The area of the triangle $= \frac{b-a}{2} \cdot \frac{b-a}{2} M = \frac{(b-a)^2}{4} M$, while the left side gives the area of the curve AB. Hence the result.

2° Let us remove the restriction $f(a)=f(b)=0$. The natural generalization that geometry yields is:

$$|J| \equiv \left| \int_a^b f(x)dx - \tfrac{1}{2}(b-a)\left\{f(a)+f(b)\right\} \right| < \frac{M(b-a)^2}{4} - \frac{1}{4M}\left\{f(b)-f(a)\right\}^2.$$

Proof:—

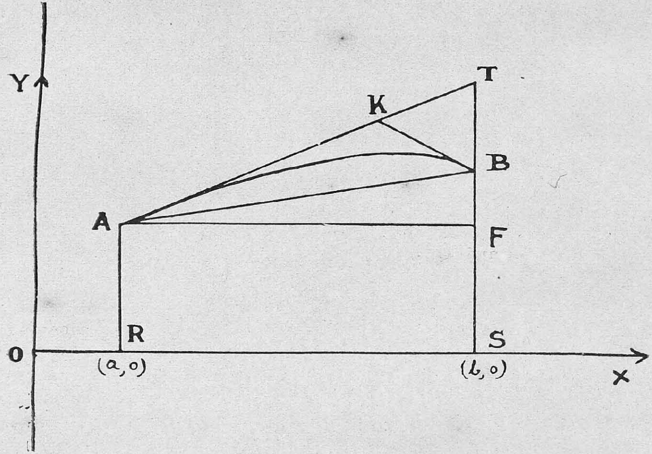

FIG. I

Here as before KA, KB are equally inclined to the x-axis, and $\tan \alpha = M$, $\alpha = K\hat{A}F$, so that KTB is isosceles, KT KB being equally inclined to TB.

The figure shows that trapezium $ARSB = \frac{b-a}{2}\left\{f(a)+f(b)\right\}$;

$\therefore$ | J | $\equiv$just the area between the curve AB and the chord AB and this is less than the area of the triangle KAB.

Now, $\triangle$ KAB $= \triangle AFT - \triangle AFB - \triangle KTB.$

$$= \tfrac{1}{2}\ AF \cdot FT - \tfrac{1}{2}\ AF \cdot FB - \tfrac{1}{2}\ TB \cdot \tfrac{1}{2}\ TB \tan KTB$$

$$= \tfrac{1}{2}\ (b-a)^2 \tan \alpha - \frac{b-a}{2}\Big[f(b)-f(a)\Big] - \frac{TB^2}{4} \cot \alpha$$

$$= \frac{(b-a)^2 \tan \alpha}{2} - \frac{b-a}{2}\left\{f(b)-f(a)\right\}$$

$$-\frac{1}{4}\Big[(b-a) \tan \alpha - \left\{f(b-f(a)\right\}\Big]^2 \cot \alpha$$

$$= \frac{(b-a)^2}{4} \tan \alpha - \frac{\{f(b)-f(a)\}^2}{4} \cot \alpha$$

$$= \frac{(b-a)^2}{4}\ M - \frac{1}{4M}\left\{f(b)-f(a)\right\}^2.$$

3° To prove that

If $|\phi''(x)| \leqslant M$ in $(a \leqslant x \leqslant b)$ and $\phi(a)=\phi(b)=0$

then $|\phi(x)| \leqslant \frac{M(b-a)^2}{8}$.

Proof:—Write $\phi(x) = \int_a^x f(x)dx.$

so that the problem becomes :—

If $|f'(x)| \leqslant M$ in $(a \leqslant x \leqslant b)$ and $\int_a^b f(x)dx = 0$

then $\left|\int_a^x f(x)dx\right| \leqslant \frac{M(b-a)^2}{8}.$

[*Note*:—There is *no* condition that $f(a)=f(b)=0$]

$\because \int_a^b f(x)dx = 0$, the curve $y=f(x)$ must cross the x-axis at least once in (a, b) say at L, so that the positive area and the negative area exactly, balance each other. Let R, S be the points $(a, 0)$ and $(b, 0)$ and let A, B be the points $\{a, f(a)\}$ and $\{b, f(b)\}$ respectively. Since $|f'(x)| < M$ it is possible to draw a line PLQ making an angle $\tan^{-1} M$ with RS and meeting the ordinates $x=a$ and $x=b$ at P and Q so that the two areas are included in the triangles PRL and LSQ.

Now $\left|\int_a^x f(x)dx\right| <$ area under AL = area under BL numerically.

And obviously

area under AL $< \triangle$ PRL,

area under BL $< \triangle$ QSL.

$$\therefore \quad \left| \int_a^x f(x)dx \right| < \text{smaller of the two triangles.}$$

i.e. $< \triangle$ QSL.

Now, the *smaller* triangle QSL must have its base $SL \leqslant \frac{1}{2}(b-a)$.

$$\therefore \quad SQ < \tfrac{1}{2}(b-a)M.$$

$$\therefore \quad \triangle QSL < \frac{(b-a)^2}{8}M.$$

$$\therefore \quad \left| \int_a^x f(x)dx \right| < \frac{(b-a)^2}{8} M.$$

4° To prove that

$$\left| f(x) - f(a) - \frac{x-a}{b-a} \{f(b) - f(a)\} \right| < \frac{M(b-a)^2}{8}$$

where $f''(x) \leqslant M$.

Notice that if we write $|\phi(x)|$ for the left hand side then $\phi''(x)$ is exactly $f''(x)$. Thus this example is exactly the last one in 3° in a disguised form.

5° If $|\phi''(x)| \leqslant M$ in $a \leqslant x \leqslant b$ and $\phi(a) = \phi(b) = 0$, and $\phi'(a) = \phi'(b) = 0$ then

$$|\phi(x)| < \frac{M(b-a)^2}{16}.$$

Proof:—Write $\phi(x) = \int_a^x f(x)\,dx$ so that we have to prove that $\left| \int_a^x f(x)\,dx \right| < \frac{M(b-a)^2}{16}$, $f'(x) < M$; it being given that $f(a) = f(b) = 0$ and $\int_a^b f(x)dx = 0$.

The condition implies that the area above the x-axis is equal to that below it.

[NOTE:—The point P in Fig. 2 where the curve crosses AB need not be the middle point of AB, though it appears as such.]

The area under the curve AP is equal and opposite to that under the curve PB,—one being positive, the other negative.

Obviously $\int_a^x f(x)\,dx <$ area under AP. Now the area under AP $<$ isosceles Δ AQP and the area under PB $<$ isosceles Δ PRB and since in magnitude both areas are equal,

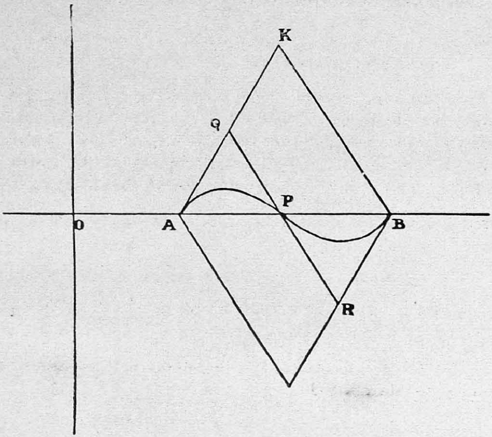

FIG. 2

$\therefore$ area under AP $<$ the smaller of the two isosceles triangles.

The base of the smaller triangle does not exceed $\frac{1}{2}$ AB.

$\therefore$ The area $< \frac{1}{16}(b-a)^2 \tan\alpha$ where $\alpha = K\hat{A}B$.

Thus $\left| \int_a^x f(x)\,dx \right| < \dfrac{(b-a)^2}{16}$ M.

Note that the result is true even if the curve crosses AB several times.

Fergusson College, Poona. G. S. MAHAJANI.

ASTRONOMICAL NOTES

Planet Notes From January to March 1939

All times given here are I. S. T. and the co-ordinates of planets given are for 5 hours 30 minutes I. S. T.

Sun. The Sun after passing the Winter Solstice on 22nd December 1938, moves with decreasing South Declination till 21st March at which time the Sun reaches the vernal equinox. The beginning of Spring indicates the passing of the Sun from South to North of the equator and it moves with an increasing North declination.

The Sun's position at the beginning of each month will be as follows :—

	1st January	*1st February*	*1st March*
R. A. ...	18 h 42 m	20 h 55 m	22 h 44 m
Decl. ...	23° 6′S	17° 25′S	8° 0′ S.

Moon. The following phases of the Moon will occur during the period January to March 1939.

	Full Moon	*Last Quarter*	*New Moon*	*First Quarter*
January	6 : 3- 0 A.M.	12 : 6-40 P.M.	20 : 6-57 P.M.	28 : 8-30 P.M.
February	4 : 1-25 P.M.	11 : 9-42 A.M.	19 : 1-58 P.M.	27 : 8-56 A.M.
March	5 : 11-30 P.M.	13 : 3-7 A.M.	21 : 7-19 A.M.	28 : 5-46 P.M.

The Moon will be nearest the earth on 6th January, 4th February and 4th March and it will be farthest away on 21st January, 17th February and 16th March.

Moon's positions.

	1st January	*1st February*	*1st March*
R. A. ...	2 h 5 m	5 h 36 m	6 h 15 m
Decl. ...	13° 53′ N.	20° 2′ N.	19° 20′ N.

Mercury. Mercury moves with a direct motion till 24th March when it will be stationary and changes its direction. During the early part of January the planet is visible as an evening object for about an hour after sunset and later it is too near the Sun to be seen. Mercury will be at its greatest elongation West and East on

3rd January and 17th March, and will be in superior conjunction with the Sun on 19th February. It will also be in conjunction with the Moon on 18th January and 22nd March.

Positions of the planet:—

	1st January	*1st February*	*1st March*
R. A. ...	17 h 5 m	20 h 5 m	23 h 18 m.
Decl. ...	20° 39′ S	22° 0′ S	5° 38′ S.

Venus. During the period January to March Venus is a morning object rising about 3 hours before the Sun and its motion is direct throughout. Venus will be at its greatest elongation west on January 30th, and will be in conjunction with the Moon on 16th January, 15th February, and 17th March.

The positions of the planet will be as follows:—

	1st January	*1st February*	*1st March*
R. A. ...	15 h 44 m.	17 h 36 m.	19 h 46 m.
Decl. ...	15° 37′ S	19° 48′ S	19° 35′ S.

Mars. This planet will be seen rising on the eastern sky after midnight. It rises earlier day by day and by the end of March it will rise by midnight. It will be in conjunction with the Moon on 14th January, 12th February and 13th March.

The positions of the planet will be as follows:—

	1st January	*1st February*	*1st March*
R. A. ...	14 h 41 m.	15 h 58 m.	17 h 9 m.
Decl. ...	14° 42′ S	19° 48′ S	22° 36′ S.

Jupiter. During the early part of January Jupiter will be visible as an evening object for about three hours after sunset in the western sky but later it approaches the Sun and is in conjunction with it on 6th March. It will be in conjunction with the Moon on 23rd January and 20th February.

The positions of the planet will be:—

	1st January	*1st February*	*1st March*
R. A. ...	22 h 11 m.	22 h 37 m.	23 h 1 m.
Decl. ...	12° 16′ S	9° 50′ S	7° 19′ S.

Saturn. Saturn will transit about sunset on the 1st January and by the end of March, Saturn is very close to the Sun. This planet will be in conjunction with the Moon on 26th January, 23rd February and 22nd March.

The positions of the planet will be :—

	1st January	*1st February*	*1st March*
R A. ...	0 h 46 m.	0 h 52 m.	1 h 2 m.
Decln. ...	2° 15′ N.	3° 3′ N.	4° 10′ N.

Uranus. The planet will be stationary on the 22nd January and its position at the beginning of each month will be as follows :—

	1st January	*1st February*	*1st March*
R. A. ...	2 h 47 m.	2 h 46 m.	2 h 48 m.
Decl. ...	15° 42′ N.	15° 40′ N.	15° 51′ N.

Neptune. Neptune will be in opposition with the Sun on 13th March and its position will be as follows :—

	1st January	*1st February*	*1st March*
R. A. ...	11 h 37 m.	11 h 36 m.	11 h 34 m.
Decl. ...	3° 43′ N.	3° 53′ N.	4° 9° N.

Popular Astronomy gives an interesting announcement of the discovery of two more new satellites of Jupiter (Jupiter X and Jupiter XI). These two satellites were discovered at the Mount Wilson Observatory by Seth B. Nicholson on photographs made with the 100 inch telescope. Both are of the 19th magnitude and hence they can be seen only through powerful telescopes.

Trivandrum. H. SUBRAMANI AIYAR.

ANNOUNCEMENTS AND NEWS

Dr. Ram Behari and Dr. R. Vaidyanathaswami have been elected Hon. Secretaries of the Indian Mathematical Society for 1938-40.

The International Congress of Mathematicians

On invitation by the American Mathematical Society, an International Congress of Mathematicians will be held in Cambridge, Massachusetts, in 1940, the dates being September 4-12.

Former Congresses. In connection with the World's Columbian Exposition in Chicago in 1893 there was an International Congress of Mathematicians. The first congress in Europe was held at Zurich in 1897, and the most recent one met in Oslo in 1936. In the interim, with a few omissions necessitated by the World War, there have been sessions about once every four years. These have all been in Europe except for that of 1924, which was held at the University of Toronto.

It is expected that the Society will be able to furnish room and board without charge for a considerable number of foreign guests during the week that they are in Cambridge.

Following precedent, there will be a score of invited addresses, each of an hour's length, and sectional meetings for the presentation of short papers.

An innovation will be conferences, somewhat after the pattern of recent international gatherings in Moscow for Topology and in Zurich for Probability, each devoted to some field in which a vigorous advance has recently been made or is in progress. Among the conferences will be one on Algebra, with Professor A. A. Albert as chairman, one on the Theory of Measure, Probability, and Allied Topics, with Professor Norbert Wiener as chairman, one on Mathematical Logic with Professor H. B. Curry as chairman, and one on Topology with Professor Solomon Lefschetz as chairman.

Six sections are tentatively planned for the presentation of papers: (i) Algebra and Theory of Numbers; (ii) Analysis; (iii) Geometry and Topology; (iv) Probability, Statistics, Actuarial Science, Economics; (v) Mathematical Physics and Applied Mathematics; (vi) Logic, Philosophy, History, Didactics. The International Commission on the Teaching of Mathematics proposes to have a session in connection with the Congress.

Membership. Membership in the Congress will be open to all persons, whether they are able to be present in person or not. For regular members of the Congress, the fee is $ 10.00; these persons may communicate papers and will receive the Proceedings of the Congress.

Information. Those interested in receiving information may file their names in the Office of the Society, and such persons will receive from time to time information regarding the program and arrangements.

Communications should be addressed to the American Mathematical Society, 531 West 116th Street, New York City, U.S.A.

(*Adapted from the Bull. of the American Mathematical Society, September, 1938*)

As usual Christmas has been a period of group activities. The Indian Science Congress met in Lahore under the Presidentship of Dr. J. C. Ghosh of the Dacca University. The Mathematics and Physics section was presided over by Dr. K. R. Ramanathan of the meteorological office, Poona. His Presidential address dealt with "The Earth's Magnetism and the upper atmosphere."

The Second Indian Statistical Conference met at Lahore, with Dr. T. E. Gregory, Economic Advisor to the Government of India as President. The Presidential Address dealt with the broad problems of the relation between the Statistician and the Community.

The Indian Academy of Sciences celebrated its Fourth Session at Madras under the Presidency of Sir C. V. Raman, its President-Founder. This meeting was noteworthy as it synchronised with the Fiftieth Birthday of Sir C. V. Raman and the completion of ten years of research on the Raman Effect. An address and a special commemoration number of the Proceedings of the Academy were presented to him on the occasion. The Indian Mathematical Society of which Sir Raman is an Honorary Member joins in wishing him many long years of health and usefulness.

Mr. R. S. Varma of the Christ Church College, Cawnpore, has been awarded the degree of Doctor of Science of the Allahabad University. The subject of his thesis was "Parabolic Cylinder Functions."

Dr. Ram Behari, Reader in Mathematics, Delhi University, has been elected a Fellow of the National Academy of Sciences of India.

THE LATE PANDIT HEMRAJ

PANDIT HEMRAJ, Principal, Dyal Singh College, Lahore, was born in Daffar, a village in the Hoshiarpur District of the Panjab on the 22nd November, 1885. His father Pt. Ganga Ram, who survives him, was a poor Brahmin, and it was under hard circumstances, therefore, that Pandit Hemraj received his education. He passed his middle school examination from the S. D. School, Hoshiarpur, in 1903, and his matriculation from the Govt. High School, Hoshiarpur, in 1905. He passed his B. A. and M. A. examinations from the D. A. V. College, Lahore with great distinction standing first in the first class in the M. A. examination in 1912. L. Chamba Mal Sud of Hoshiarpur rendered him considerable financial help during these years.

Shortly after passing his M. A., he was appointed in October 1912, an assistant Professor of Mathematics in the Dyal Singh College, Lahore. In 1916, Pandit Hemraj distinguished himself by successfully solving questions set by Professor Grace of Cambridge, who had been invited by the University of the Panjab to deliver a course of extension lectures. Panditji's work thus came to the notice of the Panjab Government and Sir John Maynard, the then Vice-Chancellor of the Panjab University, moved the University to offer Pandit Hemraj a scholarship to go abroad for higher studies. The offer was not accepted for reasons of health. By hard work Panditji rose to the Principalship of the College in 1921, and served it ably till the time of his death on the night between the 12th and the 13th November, 1938, at some minutes past midnight. In 1927, when Dr. S. S. Bhatnagar devised a Magnetic Interference Balance which has now become well-known all the world over, the Mathematical analysis of the experimental results proved a serious problem. Dr. Bhatnagar consulted several mathematicians in India and abroad. The problem was finally solved by Pandit Hemraj. Panditji's death has left a void in the academic life of the Panjab University as also in mathematical India.

Panditji was extremely simple in habits, very kind to his pupils and yet a good disciplinarian, and was in peace with every one. His life was a great lesson to all who were privileged to study at his feet. The simple village folk who knew him from his early school days as classmates or otherwise did not go to the plough when the news of his death was received. Panditji leaves behind him two sons and a small daughter besides a widow. Panditji was a prominent member of the Indian Mathematical Society and was for several years a member of the executive body. He published only a small part of his work. If his personal library and note-books are looked into, I feel sure that some work of remarkable originality may be found.

Government College,
Hoshiarpur.

HANSRAJ GUPTA.

BOOKS RECEIVED FOR REVIEW

JAHNKE-EMDE: *Functionentafeln mit Formeln und Kurven*, 3rd (revised) Edition, 1938, B. G. Teubner pp. 305 Price R. M. 15 (less 25% abroad)

NATHAN LAZAR: *The importance of certain concepts and laws of logic for the study and teaching of geometry*, New York, 1938 pp. 66 price $ 1.00.

KEYSER, SMITH, KASNER, RAUTENSTRAUCH (Addresses) *Scripta Mathematica Forum Lectures* Yeshiva College, New York, 1937, pp. 94.

G. H. HARDY and E. M. WRIGHT: *Introduction to the Theory of Numbers*, Oxford Univ. Press. 25 s.

R. A. FRAZER, W. J. DUNCAN, and A. R. COLLAR: *Elementary matrices and some applications to Dynamics and Differential equation*: Camb. Univ. Press 1938, pp. 416.

MILNE-THOMSON: *Theoretical Hydrodynamics*; Macmillan, 1938, pp. 552, Price 31 s. 6d.

REVIEWS

NATHAN LAZAR: *The importance of certain Concepts and Laws of Logic for the Study and Teaching of Geometry.*

The Mathematics Teacher, 525 W. 120th St. N.Y.C. 1938 pp. 66 price $ 1.00.

This booklet deals with the logical bases of elementary theoretical geometry, the sources of study being ninety-three modern text-books besides the monumental volumes on Euclid's *Elements* by Heath and the classical works of that eminent logician, mathematician and educator, De Morgan whose strong insistence on the necessity of proper appreciation of certain logical concepts in the study of the elements of geometry did much to influence the technique of the teaching of the subject.

The author examines critically the traditional definitions of the concepts of the Converse and the Inverse and the laws of Contraposition and of Converses with special reference to elementary geometry and attempts to show that they are to a certain extent defective in statement and restrictive in scope. Alternative definitions which are wider are proposed and illustrated and their effect on the definitions of 'Necessary and sufficient Conditions' and on the methods of proof in loci problems is noticed.

Briefly the modifications suggested are as under:—To obtain the Converse of a proposition, interchange any number of Conclusions with an equal number of data. To obtain the inverse, break up the proposition into subsidiary propositions each with *one* conclusion and in each of these latter, contradict *one* of the data and also the conclusion. To obtain the contrapositive break up the proposition again as before and interchange the Contradictory of *one* of the hypotheses and that of the conclusion.

The author holds that since educational psychology points to evidence against the possibility of automatic transfer of training from one subject to another unless 'identical elements' are present in both subjects and since, transfer being a form of generalisation, training can be of such a type as to encourage or hinder such transfer, 'it is possible to utilise geometry as a medium for making the pupils conscious of the existence of logical patterns of valid and invalid reasoning in mathematics as well as in the thinking of everyday life.' He concludes with a few pedagogical recommendations and a copious bibliography of relevant literature.

Altogether, the book makes a stimulating and instructive reading and deserves careful attention with a view to ascertaining how far it would be possible to implement the suggestions in actual teaching in the High School.

Training College, Rajahmundry.
K. SATYANARAYANA.

G. VALIRON: *Sur les Valeurs Exceptionelles des Fonctions Mèromorphes et de leurs dèrivees,* (Actualités Scientifiques et Industrielles, No. 570, price 18 frs.)

This booklet is devoted to a study of some of the recent developments in the Theory of Normal Families of Meromorphic functions. A family or class of functions Meromorphic in a domain D is said to be normal in D when any sequence of functions selected from the family contains a sub-sequence converging uniformly in any closed region interior to D. The significance of this definition will become clear if we recall that any sequence of numbers (real or complex) contains a sub-sequence converging to a limit (finite or infinite). This property does not hold, in general, for classes of elements other than numbers, even though convergence and limits could be defined for such classes. So it becomes important to consider classes of elements for which the above property is postulated. If we consider a family of functions meromorphic in a domain D and if we replace ordinary convergence of numbers by uniform convergence of functions in D and postulate for this class of functions the property specified above we are led to the idea of a normal family.

A short account of the chief criteria for normal families is found in the first chapter of the booklet under review. The connection between exceptional values and normal families is found in the fact that a family of functions regular in D is normal in D if every function of the class omits to take two specified values in D. The result is true for a family of meromorphic functions if we postulate three exceptional values. Instead of two exceptional values for the functions of a family regular in D, if we assume that no function of the class takes a specified value while the kth derivatives of the functions of the family omit to take another specified value, can we say that the family is normal? The answer is in the affirmative if the exceptional value of the kth $(k \geq 1)$ derivatives be different from zero. A proof of this proposition is found in the second chapter. Extensions and generalisations of this last result for regular functions and corresponding results for meromorphic functions are discussed in the remaining chapters. References to original memoirs and other sources are found at appropriate places in the footnotes.

V. GANAPATI IYER.

50022

Yugoslavia

Publications Mathematiques d' Universite de Belgrade.

Italy

1. Rendiconti del circolo Matematico de Palermo.
2. Rend. del Seminario matem. Univ. Padova.
3. Rend. del Seminario matem. Univ. Roma.
4. Rend. dell Accademia delle Scienze, Napoli.
5. Annali di Matematica Pura ed Applicata.
6. Atti della Reale Accademia Nazionale dei Lincei.

Spain

1. Revista Mathematica Hispano-Americana, Madrid.
2. Matematica Elemental.

Africa

Transactions of the Royal Society of South Africa.

U. S. A.

1. American Mathematical Monthly.
2. The Mathematics Teacher.
3. Popular Astronomy.
4. Duke Mathematical Journal.
5. American Journal of Mathematics.
6. Annals of Mathematics.

South America

Publications of the Universidad Nacional de la Plata.

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